QUANTUM DESIGNS

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Gerhard Zauner

Motivation (MUBs)

Let $\mathbf{A}_i$, $1 \leq i \leq k$ be k self-adjoint $d \times d$ matrices (observables) with spectral decompositions

$$\mathbf{A}_i = \sum_{l=1}^{g_i} a_{il} \mathbf{P}_{il}$$

Within *Quantum Probability Theory* the matrices are pairwise independent (w.r.t. $\frac{1}{d}\mathbf{I}$) if

$$\operatorname{tr}(\mathbf{P}_{il}\mathbf{P}_{jm}) = \frac{1}{d} \operatorname{tr}(\mathbf{P}_{il}) \operatorname{tr}(\mathbf{P}_{jm})$$

for all $1 \leq i \neq j \leq k$ and $1 \leq l \leq g_i$, $1 \leq m \leq g_j$.

In case the projection matrices are *regular*, i.e. $\operatorname{tr}(\mathbf{P}_{il}) = r$ for all $1 \leq i \leq k$, $1 \leq l \leq g$ it follows

$$\begin{aligned} k &\leq \frac{r(d^2-1)}{d-r} \\ k &\leq d+1 \qquad \text{for } r = 1. \end{aligned}$$

The last case corresponds to the definition of *mutually unbiased bases* [3], whereas the the case $r > 1$ defines a generalization towards *affine quantum designs* as given in [5]. Also here solutions reaching these bounds exist whenever d is a powers of a prime.

The infinite case

This definition (for arbitrary rank Eigenspace-projections) also allows a generalization to infinite dimensional Hilbert Spaces:

Let $\mathbf{A}$ and $\mathbf{B}$ be two self-adjoint operators over a seperable Hilbert-Space with spectrum $\sigma(\mathbf{A})$, $\sigma(\mathbf{B})$ respectively. Let further χ_E, χ_F be the characteristic functions of a Borel subset E, F of the spectrum of $\mathbf{A}$ and $\mathbf{B}$, hence $\chi_E(\mathbf{A})$, $\chi_F(\mathbf{B})$ be spectral projections of $\mathbf{A}$, $\mathbf{B}$.

We say that $\mathbf{A}$ and $\mathbf{B}$ are independent if there are borel measures μ_A, μ_B on the spectrum of $\mathbf{A}$, $\mathbf{B}$ such that, for any two compact subset E and F of the spectra the following relation holds:

$$\operatorname{tr}(\chi_E(\mathbf{A})\chi_F(\mathbf{B})\chi_E(\mathbf{A})) = \mu_A(E)\mu_B(F).$$

It can be easy seen [5], that in the finite case this definition is equivalent to the one given above.

With the Lebesgue-Borel measure on the real number, the position operator $\mathbf{X}$ and the impuls operator $\mathbf{P}$ are e.g. mutually independent, as well as linear-combinations of these two.

The general definition

A quantum design is a set $\mathbf{D} = \{\mathbf{P}_1, \ldots, \mathbf{P}_v\}$ of $v \geq 2$ orthogonal $d \times d$ projection matrices.

- $\mathbf{D}$ is said to be **regular** if there exists an $r \in \mathbb{N}$ such that

  $$\operatorname{tr}(\mathbf{P}_i) = r \qquad \text{for all } 1 \leq i \leq v.$$

- The **degree s** of $\mathbf{D}$ is given by the cardinality of the set

  $$\Lambda = \{\operatorname{tr}(\mathbf{P}_i\mathbf{P}_j) : \, 1 \leq i \neq j \leq v\}.$$

- A subset of $\mathbf{D}$ is an **orthogonal class**, if its projections are pairwise orthogonal. If the sum of all its projections adds up to the identity matrix, the orthogonal class is said to be **complete**. A quantum design is called **resolvable** if it can be written as a disjoint union of complete orthogonal classes.

- A regular resolvable quantum design with degree 2 (i.e. $\Lambda = \{0, \lambda \neq 0\}$) will be called an **affine quantum design**.

Combinatorial Designs

For classical=**commutative** quantum design (out of pairwise commuting projections) these definitions correspond to those of classical **combinatorial design theory** (see [1] and [2]).

The assignment goes via the 0/1 diagonal entries of the simultaneously diagonalized projection matrices taken as *incidence functions* of subsets of a set of d elements.

EXAMPLE: The incidence matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

corresponds to the unique *projective plane* of order 2. It is associated (e.g. by taking the columns as diagonal entries of 7 rank-3 Projections) with a commutative quantum design that is regular (with $r = 3$), has degree 1 (with $\lambda = 1$) and fulfills $\sum_{i=1}^{7} \mathbf{P}_i = 3\mathbf{I}$.

SIC POVMs

SIC POVMs (see [4]) are (in the language of quantum designs) regular ($r = 1$) quantum designs with degree 1 and the maximum of d^2 elements. It is assumed that solutions exist in any dimension d.

The same upper bound d^2 applies for all regular degree 1 quantum designs with abitrary rank. Until now no systematic study of these designs for higher ranks has been done.

2-designs

SIC POVMs, as well as complete sets of MUBs, are also so called 2-designs, which means, that in the following formula equality applies:

$$\frac{1}{v^2} \sum_{i=1}^{v} \sum_{j=1}^{v} \left(\operatorname{tr}(\mathbf{P}_i \mathbf{P}_j)\right)^2 \geq \frac{2}{d(d+1)}.$$

Also this formula can be generalized to arbitrary regular quantum design:

$$\frac{1}{v^2} \sum_{i=1}^{v} \sum_{j=1}^{v} \left(\operatorname{tr}(\mathbf{P}_i \mathbf{P}_j)\right)^2 \geq \frac{r^2(dr^2 + d - 2r)}{d(d^2 - 1)}.$$

with equality exactly when the quantum design is a 2-design (which e.g. applies to all affine quantum designs, that reach the bounds given above).

References

[1] Beth, Th., Jungnickel, D. und Lenz, H.: *Design Theory*, BI-Wissenschaftsverlag, Mannheim, Wien, Zürich, 1985.

[2] Colburn, C.J. and Dinitz, J.H.: *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, FL, 1996.

[3] Durt, T., Englert, B.G., Bengtsson I. and Zyczkowski, K.: On mutually unbiased bases, arXiv:1004.3348,2010

[4] Scott, A.J. and Grassl, M..: SIC-POVMs: A new computer study, arXiv:0910.5784, 2009

[5] Zauner, G.: *Quantendesigns / Quantum Designs*, Ph.D.Thesis, Wien / Vienna, 1999[1].

[1]see www.gerhardzauner.at, also to be published in IJQI