Mutually Unbiased Constellations

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Mutually unbiased constellations in $\mathbb{C}^d$ consist of vectors which are either pairwise orthogonal or have a squared overlap of size $1/d$. They have been introduced to investigate the existence of complete sets of mutually unbiased bases in spaces of dimensions not equal to a prime power. Two results obtained by using mutually unbiased constellations lines are reviewed.

I. INTRODUCTION

Given a quantum system of dimension d, a *complete set of mutually unbiased (MU) bases* in $\mathbb{C}^d$ consists of $d(d+1)$ pure states $|\psi_j^b\rangle$, $b=0,1,\ldots,d$, $j=1,\ldots,d$, which satisfy the conditions

$$\left|\langle\psi_j^b|\psi_{j'}^{b'}\rangle\right| = \begin{cases} \delta_{bb'} & \text{if } b=b' \,, \\ \frac{1}{\sqrt{d}} & \text{if } b\neq b' \,. \end{cases} \qquad (1)$$

Thus, the states form $(d+1)$ orthonormal bases, and scalar products between states taken from different bases have constant modulus. If the dimension d equals a power of a prime, complete sets of MU bases do exist. It is not possible to have more than $(d+1)$ such bases. For *composite* dimensions $d=6,10,12,\ldots$, however, the maximum number of MU bases poses an open problem despite many efforts reviewed in [1].

Some questions about MU bases are easier to study using the concept of *MU constellations.* Clearly, if a complete set of seven MU bases were to exist in $\mathbb{C}^6$, then any MU constellation obtained by removing some of these 42 vectors would exist as well.

Motivated by this observation, we define *constellations of MU vectors* in $\mathbb{C}^d$ as $(d+1)$ sets of x_b pure states $|\psi_j^b\rangle$, $b=0,1,\ldots,d$, $j=1,\ldots,x_b$, which satisfy the conditions (1). The $(d+1)$ integers $x_b\in\{0,\ldots,d-1\}$ specify all possible types of MU constellations which will be denoted by $\{x_0,x_1,\ldots,x_d\}_d$. If the number x_b in a MU constellation equals zero, it corresponds to an *empty* set and will be suppressed. For example, $\{2,1,2,0\}_4\equiv\{2,1,2\}_4$ denotes a MU constellation in $\mathbb{C}^4$ which consists of two pairs of orthonormal vectors and one single vector. Since the ordering of the bases within a constellation is irrelevant, we arrange them in decreasing order, using the shorthand x^a if there are a bases with x elements: $\{2,1,2\}_4$ thus becomes $\{2^2,1\}_4$.

In the following two sections we will first describe numerical studies regarding the existence of MU constellations followed by evidence against a conjecture linking MU bases and affine planes. Both results relate to dimension $d=6$.

II. MISSING MU CONSTELLATIONS IN $\mathbb{C}^6$

Suppose you want to find the MU constellation $\{x_{0,}x_{1,\ldots,}x_d\}_d$. To do so, it is sufficient to search for it in a space $\mathcal{C}_d(x)$ which can be parameterized by p_d angles denoted by $\vec{\alpha}=(\alpha_1,\ldots,\alpha_{p_d})^T$[2]. Defining

$$\chi_{jj'}^{bb'} = \begin{cases} \delta_{jj'} \text{ if} & b=b' \,, \\ \frac{1}{\sqrt{d}} \text{ if} & b\neq b' \,, \end{cases} \qquad (2)$$

the non-negative function $F:\mathbb{R}^{p_d}\to\mathbb{R}$

$$F(\vec{\alpha}) = \sum_{1\le b\le b'}^{d}\sum_{j=1}^{x_b}\sum_{j'=1}^{x_{b'}}\left(\left|\langle\psi_j^b|\psi_{j'}^{b'}\rangle\right| - \chi_{jj'}^{bb'}\right)^2 . \qquad (3)$$

equals zero if and only if the parameters $\vec{\alpha}$ correspond to a MU constellation.

It is thus possible, in principle, to prove the (non-)existence of a MU constellation by determining whether the smallest value of the function $F(\vec{\alpha})$ is non-zero. This means to identify its (possibly degenerate) *global* minimum which, unfortunately, is not a simple numerical task: the global minimisation of a nonlinear function such as a polynomial of fourth order in sufficiently many variables may already pose a NP-hard problem [4]. A well-known strategy is to search for minima by starting from random initial points which, however, may turn out to be *local* ones. By repeating the process sufficiently often, it is likely to detect the global minima as well..

Note that the choice of the function $F(\vec{\alpha})$ is not unique. The expression (3) is convenient because efficient minimisation tools are available for a sum of squares. In particular, the Levenberg-Marquardt algorithm [5, 6], often used in Regressional Analysis, cleverly switches between the method of steepest descent and the Gauss-Newton algorithm to speed up convergence.

The numerical minimization of $F(\vec{\alpha})$ is a reliable tool to identify MU constellations since extensive searches for existing MU constellations in dimensions five and seven have been performed successfully [2].

The results of the searches performed in dimension six have been collected in Table I. Strong evidence emerges that *not all* MU constellations of the form $\{5,x,y,z\}_6$ exist:

- the *largest existing* MU constellations are $\{5,4^2,1\}_6$ and $\{5^2,3,1\}_6$ both containing 15

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$d=6$	parameters p_6					success rate				
λ,μ	ν					ν				
	1	2	3	4	5	1	2	3	4	5
1,1	10	-	-	-	-	100.00	-	-	-	-
2,1	15	-	-	-	-	100.00	-	-	-	-
2,2	20	25	-	-	-	100.00	100.00	-	-	-
3,1	20	-	-	-	-	100.00	-	-	-	-
3,2	25	30	-	-	-	99.95	100.00	-	-	-
3,3	30	35	40	-	-	99.42	39.03	0.00	-	-
4,1	25	-	-	-	-	100.00	-	-	-	-
4,2	30	35	-	-	-	92.92	44.84	-	-	-
4,3	35	40	45	-	-	12.97	0.00	0.00	-	-
4,4	40	45	50	55	-	0.74	0.00	0.00	0.00	-
5,1	30	-	-	-	-	95.40	-	-	-	-
5,2	35	40	-	-	-	76.71	10.96	-	-	-
5,3	40	45	50	-	-	1.47	0.00	0.00	-	-
5,4	45	50	55	60	-	0.00	0.00	0.00	0.00	-
5,5	50	55	60	65	70	0.00	0.00	0.00	0.00	0.00

Table I: Success rates for searches of MU constellations $\{5,\lambda,\mu,\nu\}_6$ in dimension six, based on 10,000 randomly chosen initial points in the allowed p_6-dimensional space.

mutually unbiased states (ignoring $\{5^3\}_6$ i.e. three MU bases with 18 states);

- the *smallest non-existing* MU constellations are $\{5,3^3\}_6$ and $\{5,4,3,2\}_6$ each consisting of 14 states;

Overall, we have been able to positively identify only 18 out of 35 MU constellations in dimension six. The numerical data strongly suggests that the 15 unobserved MU constellations do not exist. Thus, the existence of four MU bases is highly improbable.

III. MU CONSTELLATIONS AND AFFINE CONSTELLATIONS IN $\mathbb{C}^6$

The second application of MU constellations in dimension six enables us to conclude that their existence properties do not match those of *affine constellations*. Thus, a deeper relation between the existence problems of complete sets of MU bases and finite affine planes is unlikely [3].

Finite affine planes [7] are geometric structures which consist of a finite number of *points* satisfying the following postulates: (i) any two points determine a unique line; (ii) given a line and a further point, there is a unique line through this point disjoint from the given line. Trivial realizations of finite affine planes are excluded by the requirement that (iii) there exist four points such that no three of them are located on a single line. The *order d* of an affine plane is given by the number of points on each line, and the entire plane can be *foliated* into d parallel (i.e. non-intersecting) lines in $(d+1)$ different ways.

The possibility of a link between the existence problems for MU bases and affine planes has been voiced repeatedly [8, 9], with Wootters suggesting an explicit correspondence [10, 11], namely that *parallel lines of an affine plane should correspond to operators projecting on orthogonal quantum states.* Consequently, foliations are associated with orthonormal bases. Saniga, Planat, and Rosu [12] (cf. also [13]) have elevated the link between MU bases and affine planes to a conjecture:

> *"Non-existence of a projective plane of the given order d implies that there are less than $d+1$ mutually unbiased bases (MUBs) in the corresponding $\mathcal{H}^d$*, and vice versa."

A *projective* plane of order d turns into an affine plane of the same order if one line is discarded, thus covering the original claim.

Let us introduce the main concept needed for our argument, defined in analogy to MU constellations: an *affine constellation* $\langle x_0, x_1, \ldots, x_d\rangle_d$ of order d consists of $(d+1)$ sets of $x_b \in \{0,1,\ldots,d-1\}$ lines with d points each such that (a) any two lines within each set do not intersect and (b) any two lines from different sets have exactly one point in common. This notion is easily understood by example. Given an affine plane of order $d=3$, there are four different ways to arrange all nine points on three non-intersecting (parallel) lines. Ignoring, say, seven of these twelve lines (one foliation, two lines of the second foliation and one each of the remaining two), we create the affine constellation $\langle 2^2,1\rangle_3$ $(\equiv \langle 2,2,1\rangle_3)$. It consists of three sets of 2, 2, and 1 lines, respectively, such that the lines within each set indeed have (a) no point in common while (b) lines belonging to different sets share exactly one point. Futhermore, it is sufficient to specify (and hence list) only two lines of a foliation as the third one is then uniquely determined.

Clearly, if an affine plane of order d exists, all affine constellations obtained by removing one line or more also exist. If, however, for a given value of d, some affine constellation is found *not* to exist, then an affine plane of order d cannot exist. Assuming that the SPR-conjecture captures a fundamental mathematical relationship between MU bases and affine planes, one would expect the properties of affine constellations to closely parallel those of MU constellations.

To see that

> *MU constellations and affine constellations do not match in dimension six*

let us consider 36 points which are known not to support an affine plane of order six—the maximal number of foliations is three, not seven [14]. The largest possible affine constellation (containing three foliations) is given by $\langle 5^3,4\rangle_6$ since adding a fifth line to the last four would imply the existence of *four* foliations, a contradiction. This constellation *does* exist: two (standard) foliations of $\langle 5^3,4\rangle_6$ are given by the six horizontal and six vertical lines, and Table II makes the remaining ten lines explicit

54	2 ·	3 ·	63	11	42
1 ·	53	64	4 ·	22	31
2 ·	62	51	3 ·	44	13
61	1 ·	4 ·	52	33	24
32	41	23	14	5 ·	6 ·
43	34	12	21	6 ·	5 ·

Table II: This (incomplete) Graeco-Latin square represents one foliation and four additional lines of $\langle 5^3, 4\rangle_6$, the maximal affine constellation of order 6 (see text for details)

using the notation of a *Graeco-Latin square* for a pair of *mutually orthogonal Latin squares*, or MOLS [7].

The first integer in each square indicates one of the six lines of the (non-standard) foliation to which the corresponding point belongs. These integers are different in each row and column ensuring that each line has only one point in common with the standard foliations consisting of horizontal and vertical lines, respectively. Four more lines are defined by the second integers which, again, they do not repeat within any line or column. Finally, no two squares contain the same two-digit number to ensure that each of the four lines intersects those of the third foliation in a single point only.

Thus, having exhibited explicitly the affine constellation $\langle 5^3, 4\rangle_6$, we indeed observe a clash since the MU constellation $\{5^3, 4\}_6$ appears not to exist according to the numerical approach presented in Sec. II. If there is a link between MU bases and affine planes, it must be of a very subtle nature to circumvent this discrepancy.

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