SIC POVMs and the Discrete Affine Fourier Transform (the Linear Canonical Transform)

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Most examples of SIC POVMs that are known are Weyl-Heisenberg covariant. That is, using Weyl-Heisenberg operators (in the standard basis of $\mathbb{C}^d$)

$$\begin{aligned} X|n\rangle &= |n+1 \bmod d\rangle, \\ Z|n\rangle &= \omega^n|n\rangle; \quad \omega = e^{2i\pi/d}, \end{aligned} \tag{1}$$

the operators

$$D_{\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)} = \tau^{nm}X^nZ^m; \quad \tau = -e^{i\pi/d} \tag{2}$$

can be used to map a so-called "SIC fiducial" vector into d^2 vectors whose projectors form a SIC POVM. The operators of (2) form the Generalized Pauli group $GP(d)$ (suppressing a phase which is unimportant for the present discussion). The Normalizer of $GP(d)$ is the Clifford group $C(d)$, which maps $GP(d)$ onto itself via conjugation $D \mapsto UDU^\dagger$. It can be shown that Clifford group elements can be indexed by a 2x2 matrix with integer mod d entries (or mod $2d$ if d is even) and unit determinant. The conjugation then takes the form

$$U_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)} D_{\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)} U^\dagger_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)} = D_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)}, \tag{3}$$

and this is a linear map in the Weyl-Heisenberg indexes.

An interesting fact is that (almost) every SIC found has a fiducial vector that is an eigenvector to an order 3 Clifford group element. There is a conjecture (by G. Zauner [1]) that there exists a fiducial in every dimension that is an eigenvector to $U_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)}$, the so-called Zauner unitary. This gives us good reason to try to understand the Clifford group elements better. What are these things, and what do they do to quantum states? The latter is simple to formalize, but somewhat harder to understand. If β has a multiplicative inverse mod d,

$$U_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)} = \frac{1}{\sqrt{d}} \sum_{r,s=0}^{d-1} \tau^{\beta^{-1}(\alpha s^2 - 2rs + \delta r^2)}|r\rangle\langle s|, \tag{4}$$

otherwise the formula is more complicated [2]. Letting $\alpha = \delta = 0$ and $\beta = 1$ (which fixes $\gamma = -1$) we obtain the QFT. Evidently, the diagonal elements add quadratic terms to the exponent. Why is this, and what does the quadratic terms do?

It is illustrative to study the continuous-variable Fourier Transform. It can be obtained in the following way: calculate the Wigner distribution of the function, rotate the phase space so that the p axis becomes the x axis (and x becomes $-p$), and return from the phase space picture to a function by the Weyl transformation. The new function will be the Fourier Transform of the old one. It is trivial that doing this four times will return to the starting function. A *FRactional Fourier Transform* (FRFT for short) can now be defined by the same procedure, but rotating by a different angle θ. The resulting map can be represented in integral form as

$$\Big(F_\theta\psi\Big)(s) = \int e^{i\pi(\sin\theta)^{-1}\left((\cos\theta)s^2 - 2rs + (\cos\theta)r^2\right)}\psi(r)dr. \tag{5}$$

Apparently,

$$F_\theta \sim U_{\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)}. \tag{6}$$

Actually, there is a larger class of generalized Fourier Transforms that correspond to the whole Clifford group: the Linear Canonical Transforms (LCT)

$$\Big(F_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}\psi\Big)(s) = \int e^{i\pi\beta^{-1}(\alpha s^2 - 2rs + \delta r^2)}\psi(r)dr \tag{7}$$

which applies the linear transformation $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ on the phase space (real-valued entries, unit determinant). The transform is called Canonical because the unit determinant ensures that energy is not concentrated in phase space, or alternatively that the transform preserves the commutation relation. This class of transforms has been rediscovered many times, and has gained a new name (almost) every time. The names used include: Affine Fourier Transform, Generalized Fresnel Transform, Collins Transform, ABCD Transform, Almost Fourier/Fresnel Transform, Quadratic phase system, Generalized Huygens integral, Moshinsky-Quesne Transform, and more. Most of the basics was established in the 70's and 80's, but the earliest paper containing what is essentially the Fractional Fourier Transform seems to be a paper by Wiener in 1929 [3], see the textbook [4] for an overview of all this.

The Linear Canonical Transform has a direct correspondence in optics. The Fractional Fourier Transform from Eq. (6) can be obtained from propagation through graded-index media, and in addition, one can perform scaling $U_{\left(\begin{smallmatrix} A & 0 \\ 0 & 1/A \end{smallmatrix}\right)}$, chirp convolution $U_{\left(\begin{smallmatrix} 1 & B \\ 0 & 1 \end{smallmatrix}\right)}$ (propagation through free space in the Fresnel approximation) and chirp multiplication $U_{\left(\begin{smallmatrix} 1 & 0 \\ C & 1 \end{smallmatrix}\right)}$ (propagation through a thin lens). These can now be combined to give all Linear Canonical Transforms, for example $U_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)} = U_{\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)}U_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$.

Returning to finite dimension we remember that the index-matrix elements are integers mod d (or $2d$). It would seem that rigid rotations are difficult because of the integer entries in the index matrix. It would carry too far to expand on this here, but we just note that the Zauner unitary can be thought of as a rotation: it rotates the $\binom{1}{0}$ axis to the $\binom{0}{1}$ axis and then to the (inverted) diagonal $\binom{-1}{-1}$, and back to the start. While this is not a rigid rotation, it is very similar to one. One final warning to readers interested in MUBs: this "rotation" is adapted to integers mod d (or $2d$), but not to elements of a field with d elements (where such a field exists). These only coincide for d prime.

In conclusion, we note that the Clifford group really is the group of Discrete Linear Canonical Transforms.

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[1] Gerhard Zauner. *Quantendesigns: Grundzge einer nichtkommutativen Designtheorie*. PhD thesis, University of Vienna, 1999.

[2] D. M. Appleby. SIC-POVMs and the extended Clifford group. *J. Math. Phys.*, 46:052107, 2005. arXiv:quant-ph/0412001.

[3] N. Wiener. Hermitian polynomials and Fourier analysis. *J. Math. Phys. (MIT)*, 8:70–73, 1929.

[4] Haldun M. Ozaktas, Zeev Zalevsky, and M. Alper Kutay. *The Fractional Fourier Transform*. Wiley, 2001. ISBN 0471963461.