Symmetries about MUB's

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Abstract

We review the connections between three operator bases - the Pauli operators, the phase-point operators (whose averages are the discrete Wigner function), and the MUB's; and write a consistent set of transformations in operator space that relate them explicitly. We then discuss symmetries of this structure - its invariance under translations and axis permutations in phase space. These transformations induce permutations of MUB's states (of states within bases, and of bases among themselves, respectively), while preserving the full set of MUB's. We then generalize the axis permutations by introducing new basis axes in phase space, which leads to nonstandard entanglement patterns while preserving the total entanglement content of all the MUB's. Because two MUB's determine the full set of $d+1$, it follows that two different MUB sets can share at most one basis set. This talk is the subject of two papers that will be appear on the arxiv.

I. THREE OPERATOR BASES

Here we will define three operator bases, write the transformations between them, and show how the MUB's are obtained from the other two. Various aspects of these connections have been discussed by Björk and coworkers [1], Vourdas [?], and Wootters and coworkers [3].

The Pauli operators are generated in a sort of multiplication table,

$$\mathcal{O}_{n,m} = \omega^{n \cdot m/2} X^n Z^m = \omega^{n \cdot m/2} X_1^{n_1} ... X_N^{n_N} Z_1^{m_1} ... Z_N^{m_N}, \tag{1}$$

forming a $d \times d$ square array in which (if d is a prime power), the $d+1$ compatible subsets fall along rays illustrated in Figure 1. These rays are described by m/n = constant = s, where n, m, and s are elements of the finite field of order d, as in Bill Wootters' introductory talk. The shapes reflect the field multiplication rule. Rays in the Pauli table and their generalizations in the form of curves have been discussed in detail in Refs. [1] and [4],

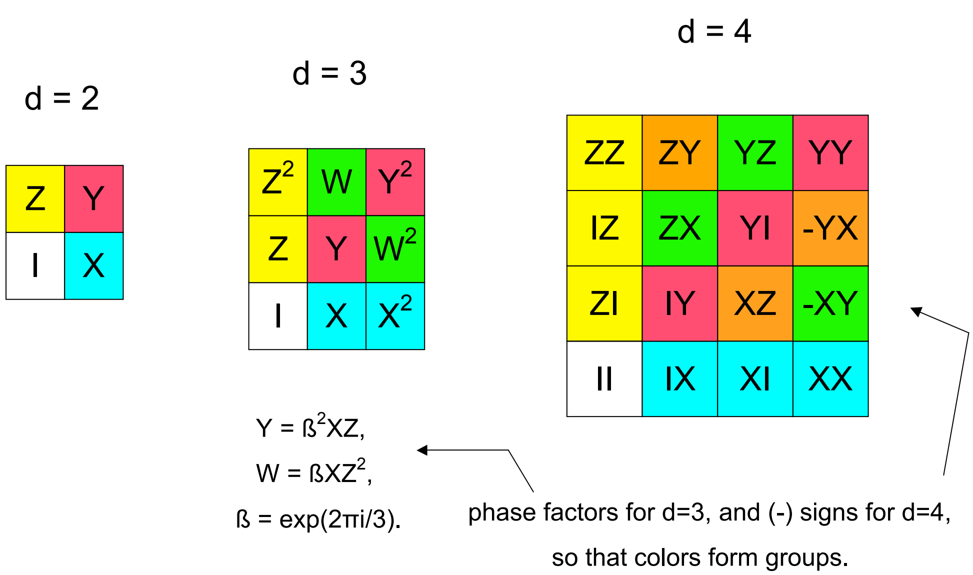

FIG. 1: Compatible operator subsets lie along the d+1 rays intersecting the origin.

The phase factors $\omega^{n \cdot m/2}$ are chosen so that the compatible subsets form groups, *ie*,

$\mathcal{O}_{n,sn} = (\mathcal{O}_{1,s})^n$. When this is done, then the MUB eigenbases are also the Fourier transforms of these compatible subsets, specifically

$$\mathcal{P}_{s,k} = \frac{1}{p}\sum_n \omega^{n\cdot k}\mathcal{O}_{n,sn}, \tag{2}$$

where the slope of the ray, s, denotes the basis and k labels the state within it. A double Fourier transform gives us the third set of operators, sometimes called the displaced parity operators,

$$\mathcal{A}_{a,b} = \frac{1}{p}\sum_n\sum_m \omega^{n\cdot b - m\cdot a}\ \mathcal{O}_{n,m}. \tag{3}$$

a relationship discussed in an extensive review of the role of Galois fields in quantum mechanics [2].

The $\mathcal{A}_{a,b}$ operators are also the phase point operators associated with the new phase space construction of Gibbons, Hoffmann and Wootters [3]. In this construction, the expectation values of the phase point operators, the discrete Wigner function, has the distinguishing feature that line sums are precisely the MUB states. Corresponding to each ray in the Pauli table, there are d parallel lines, called a *striation* in phase space [3]. The set of all $d+1$ striations, called a *foliation* of phase space, provides all of the MUB states. The *vertical* line sums are eigenstates of Z,

$$\mathcal{P}_{Z,k} = p^{-N}\sum_b \mathcal{A}_{k,b}, \tag{4}$$

where the intercept along the horizontal axis identifies the state k. The finite-slope line sums produce all other MUB states,

$$\mathcal{P}_{s,k} = p^{-N}\sum_a \mathcal{A}_{a,k+sa}, \tag{5}$$

where now, k is the intercept along the *vertical* axis. The inverse transformation expresses $\mathcal{A}_{a,b}$ as the sum over all $d+1$ lines passing through the point a,b. The explicit formula was used in Ref. [3] to construct the $\mathcal{A}_{a,b}$ directly from the MUBs themselves.

II. SYMMETRIES

1) **Displacments in phase space:** As a preliminary, we note that X is, by definition, a unit displacement operator for position, and Z, an exponential of position, is a unit displacement operator for momentum. But these refer to displacements in phase space, not

in the space where the Pauli operators live, which I shall refer to as *displacement* space. In displacement space, the same transformations show up as phase shifts. Specifically, the transformation induced by Pauli operators on other Pauli operators (which follows from their underlying commutation rule, $Z_i X_j = \omega^{\delta_{i,j}} X_j Z_i$) is

$$\mathcal{O}_{r,t}\mathcal{O}_{n,m}\mathcal{O}^{\dagger}_{r,t} = \omega^{m\cdot r - n\cdot t}\mathcal{O}_{n,m}. \tag{6}$$

It follows immediately from the Fourier transform relationship (Eq. 3) that

$$\mathcal{O}_{r,t}\mathcal{A}_{a,b}\mathcal{O}^{\dagger}_{r,t} = \mathcal{A}_{a+r,b+t}, \tag{7}$$

which represents a uniform translation in phase space through the vector (r,t). Hence the position of each Pauli operator in the multiplication table is associated with a point in displacement space. To complete this picture, the transformation of MUB states depends in a simple way on the ray $\sigma = t/r$ on which $\mathcal{O}_{r,t}$ resides. From Eqs. 2 and 6,

$$\mathcal{O}_{r,\sigma r}\mathcal{P}_{s,k}\mathcal{O}^{\dagger}_{r,\sigma r} = \mathcal{P}_{s,k+(\sigma-s)r}, \tag{8}$$

showing that MUB states undergo cyclic permutations within basis sets, except for those in the basis $s = \sigma$, which are eigenstates of $\mathcal{O}_{r,\sigma r}$ and hence invariant.

As the above equations show, the transformation produces two realizations of each operator basis that are unitarily equivalent - meaning that all elements in one basis are related uniquely to those in the other by a unitary transformation (U) in the Hilbert space, where in this case $U = \mathcal{O}_{r,t}$. Transformations that produce *inequivalent* operator bases are provided by intoducing additional s-dependent phase shifts in the groups of compatible Pauli operators, $\mathcal{O}_{n,sn} \rightarrow \omega^{l_s \cdot n}\mathcal{O}_{n,sn}$. These further transformations provide added flexibility in defining the Wigner function, and correspond to what Ref. [3] found by considering independent cyclic permutations of the MUB states outside the constraint of Eq. 8.

2) **Axis permutations:** Here we consider transformations that permute Pauli operators among rays in displacement space, or MUB states among bases (not within bases, as above). Operationally, one can imagine replacing the X and Z operators, representing the two axes of displacement space, by any other compatible subsets of operators, and regenerating the Pauli multiplication table. This operation regenerates the same full set of MUB's, following the same rays in displacement space, but with individual MUB's permuted among rays. To give a simple illustration, we fix the Z^m operator set while replacing the X^n with

$\mathcal{O}_{n,\sigma n}$, the operators that lie along the ray σ in the original Pauli table. To identify the the unitary operator that does this, first define U by the two equations $UZU^\dagger = Z$ and $UXU^\dagger = \mathcal{O}_{1,1} = \omega^{1/2}XZ$. Then U^σ accomplishes what we want:

$$U^\sigma Z^m U^{-\sigma} = Z^m \qquad \text{and} \qquad U^\sigma X^n U^{-\sigma} = \mathcal{O}_{n,\sigma n}, \tag{9}$$

and moreover, it follows that for all other operator sets,

$$U^\sigma \mathcal{O}_{n,sn} U^{-\sigma} = \mathcal{O}_{n,(s+\sigma)n}, \tag{10}$$

so that every operator (and every basis set), except for the Z's, is transformed.

To describe this transformation in more detail, note that U, like Z and X, denotes a set of N generators, so that

$$U^\sigma = U_1^{\sigma_1} U_2^{\sigma_2} ... U_N^{\sigma_N}. \tag{11}$$

The first factor permutes bases within similar groupings - product to product, and entangled to similarly entangled. Other factors permute among different groupings, product to entangled, entangled to other entangled, etc. In $d = 4$, for example, one generator interchanges the X and Y bases, and simultaneously interchanges the two Bell states. The other generator changes both X and Y bases into Bell bases, and vice-versa. In $d = 9$ there are also two generators, but each with period three. Looking at Figure 2, one can identify three groupings of bases by looking at the first column ($\mathcal{O}_{1,s}$) in the table. U_1 permutes within groupings, and U_2 permutes among groupings.

[Thinking operationally], we can imagine visiting all MUB bases in the full set of MUB's by starting out in a Z-state and Fourier transforming to X (or 0):

$$\mathcal{F}|Z,k\rangle = |0,k\rangle \qquad \text{and} \qquad U^\sigma|0,k\rangle = |\sigma,k\rangle. \tag{12}$$

The second equation represents N different kinds of operations, each cyclic with period p. One kind preserves entanglement type (including producthood), and the other kinds transform between (in general different) entanglement types.

In a more general MUB-preserving transformation, both X and Z may be replaced with any other pair from the same MUB set. This transformation is more complicated than the above, and in particular, one may have to redefine the generators of the new vertical axis in order that the transformation be unitary. But this point shows that any two MUB's from a complete set regenerate the same MUB set. As a corollary, if two MUB sets differ, they can share at most one MUB in common.

d = 9

Same definitions Y and W.

Z^2Z^2	Z^2W	Z^2Y^2	WZ^2	WW	WY^2	Y^2Z^2	Y^2W	Y^2Y^2
Z^2Z	Z^2Y	Z^2W^2	WZ	WY	WW^2	Y^2Z	Y^2Y	Y^2W^2
Z^2I	Z^2X	Z^2X^2	WI	WX	WX^2	Y^2I	Y^2X	Y^2X^2
ZZ^2	ZW	ZY^2	YZ^2	YW	YY^2	W^2Z^2	W^2W	W^2Y^2
ZZ	ZY	ZW^2	YZ	YY	YW^2	W^2Z	W^2Y	W^2W^2
ZI	ZX	ZX^2	YI	YX	YX^2	W^2I	W^2X	W^2X^2
IZ^2	IW	IY^2	XZ^2	XW	XY^2	X^2Z^2	X^2W	X^2Y^2
IZ	IY	IW^2	XZ	XY	XW^2	X^2Z	X^2Y	X^2W^2
II	IX	IX^2	XI	XX	XX^2	X^2I	X^2X	X^2X^2

FIG. 2: Pauli operators in 9X9 displacement space, with 4 product and 6 totally entangled bases. The pattern corresponds to the field F_9.

III. ENTANGLEMENT PATTERNS

All of the MUB sets shown in Figures 1 and 2, as well as those implicit in the more general discussion, are standard sets: Their multiplication table is generated by X and Z, they have $p+1$ product bases, and all other bases are totally entangled, in the sense that they are nonseparable and every particle is totally entangled with one or more other particles. The colored patterns for the $d = 4$ and 9 cases show product bases that correspond exactly to the bases of the $d = 2$ and 3 cases, respectively. These are the bases labeled by field variables in F_p. All other bases are labeled by the remaining field elements in F_{p^2}. The latter show no patterns discernible to the naked eye as straight lines. It will be shown in more detail elsewhere that all of the particles associated with these lines are totally entangled with one or more other particles. But a quick and plausible argument is that no one-particle operators appear in the Pauli operator sets - all such operators are used up in the product basis sets. By the MUB condition, all one-particle properties are completely random in these bases. This means that in a pure state of N particles, the individual particles must be totally entangled with other particles in the system.

d = 8

ZZZ	ZZY	IYZ	IYY	YIZ	YIY	XXZ	XXY
ZZI	ZZX	IYI	IYX	YII	YIX	XXI	XXX
ZIZ	ZIY	IXZ	IXY	YZZ	YZY	XYZ	XYY
ZII	ZIX	IXI	IXX	YZI	YZX	XYI	XYX
IZZ	IZY	ZYZ	ZYY	XIZ	XIY	YXZ	YXY
IZI	IZX	ZYI	ZYX	XII	XIX	YXI	YXX
IIZ	IIY	ZZY	ZXY	XZZ	XZY	YYZ	YYY
III	IIX	ZXI	ZXX	XZI	XZX	YYI	YYX

FIG. 3: Pauli operators in 8X8 displacement space showing one product, two GHZ, and 6 separable-Bell bases.

We may introduce MUB-changing transformations as generalizations of the axis permutations by replacing either or both of the X and Z axes with new compatible operator sets that do not appear in the standard MUB set. Such transformations can change its entanglement pattern - indeed this is part of the motivation for investigating these transformations. An example is shown in Figure 3, where the X^n operators of the standard MUB set have been replaced by another compatible set that is not present in the standard set. Indeed, its eigenbasis consists of separable-Bell states (SB), in which one particle separates and is in a pure state, while the other two are in a Bell state. The multipliction table thus generated has only the Z basis in common with the standard set: It consists of only that one product basis, two GHZ bases (which differ in detail from those of the standard set), and 6 SB bases (which differ in kind). One can identify the basis types by looking at the operators associated with the particular ray. Such nonstandard MUB sets have been generated in a different way, by Refs. [4] and [1], using curves rather than straight lines in the Pauli multiplication table.

There are four distinct combinations of the three possible entanglement patterns, as detailed in Ref. [5]. One of these consists of 9 SB bases [6], which is interesting because it has no MUB types in common with the standard set.

We conclude with a few remarks about nonstandard MUB sets that are taken up in more detail elsewhere:

1. In an eigenbasis of Pauli operators, all states have the same entanglement pattern.

2. One can find a compatible set of Pauli operators whose eigenbasis exhibits any given entanglement pattern. [The entanglement pattern refers here simply to the grouping of particles into nonseparable subsets, although states within the basis have all the same entanglement characteristics as well.]

3. In any eigenbasis of Pauli operators, each particle is in a state of either perfect purity, or total entanglement.

4. In a full MUB set of such eigenbases, each particle is pure in $p+1$ MUB's, and totally entangled in the remaining $p^N - p$ MUB's. [This point acts as a constraint on the combinations of entanglement patterns that can coexist in the same MUB set.]

5. Transformations like that of the last example can be made in such a way that the two MUB sets are unitarily equivalent, no matter how different their entanglement patterns.

The cases of $d = 4$ and 9 show an intimate connection between entanglement and finite field extensions, since product states are labeled by elements in F_2 and F_4, respectively, while entangled states require elements in F_4 and F_9. The last example breaks this connection, as do other nonstandard patterns that may have no product states at all. Nontheless, it may be that entanglement holds a key to a physical interpretation of the high degree of symmetry that exists in those cases where full MUB sets exist.

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