Charting the Shape of Hilbert Space

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The space of quantum states for a d-level system is usually thought of as a smooth, featureless place—simply, a linear vector space $\mathcal{H}_d$ over the complex field. But in fact, the space of quantum states corresponds to the convex set of trace-1 positive semi-definite operators *on top of* $\mathcal{H}_d$: This is a body that is anything but smooth and featureless. And in its shape may lie the key to a deeper conceptual understanding of quantum mechanics—at least this is the point of view of the Quantum Bayesian approach to quantum theory developed by Carlton Caves, Rüdiger Schack, the author, and others [1].

The reason for this is that Quantum Bayesianism, or QBism, is based on the idea that quantum theory is best understood as an *addition* to probability theory, not as a theory separate from standard probability theory or one including standard probability theory as a special case. It is an addition invoked in a situation where one wants to analyze a given physical experiment in terms of another, never-actualized (or counterfactual) experiment. As such, the first step to finding a deeper understanding quantum theory is to find a manageable representation of quantum states purely in terms of probabilities, without amplitudes or Hilbert-space operators. In this talk, we review the efforts the Perimeter Institute QBism group has made to find such a good representation.

The best candidate so far involves a mysterious entity called a "symmetric informationally complete positive-operator-valued measure," or SIC (pronounced "seek") for short. This is a set of d^2 operators $H_i = \frac{1}{d}\Pi_i$ on $\mathcal{H}_d$, where the $\Pi_i = |\psi_i\rangle\langle\psi_i|$ are rank-one projection operators such that

$$\left|\langle\psi_i|\psi_j\rangle\right|^2 = \frac{1}{d+1} \quad \text{whenever} \quad i \neq j\ . \tag{1}$$

We say mysterious because, despite 10 years of growing effort since the definition was first introduced [2, 3] (there are now nearly 50 papers on the subject), no one has been able to show that SICs exist in general finite dimensions. All that is known firmly is that they exist in dimensions 2 through 67 [4]. Dimensions 2 − 15, 19, 24, 35, and 48 are known through direct or computer-automated analytic proof; the remaining solutions are known through numerical simulation, satisfying Eq. (1) to within a precision of 10^{-38}.

What is most intriguing about a SIC is that the probabilities $P(H_i)$ for its outcomes uniquely determine the system's quantum state ρ, and it does so through an amazingly simple formula,

$$\rho = \sum_{i=1}^{d^2}\left((d+1)\,P(H_i) - \frac{1}{d}\right)\Pi_i\ . \tag{2}$$

Making us of this formula, one finds a novel way to think of the Born Rule for quantum probabilities. For instance, consider a von Neumann measurement with outcomes $D_j = |j\rangle\langle j|$ (the vectors $|j\rangle$ forming an orthonormal basis), and let $P(D_j|H_i)$ be a conditional probability for finding outcome D_j if the system had been prepared in state Π_i. Then the probability for D_j given by the standard Born Rule

$$Q(D_j) = \mathrm{tr}(\rho D_j) \tag{3}$$

becomes

$$Q(D_j) = (d+1)\sum_{i=1}^{d^2} P(H_i)P(D_j|H_i) - 1\ . \tag{4}$$

Compare this to the expression one would expect from classical probability theory (i.e., the Law of Total Probability),

$$P(D_j) = \sum_{i=1}^{d^2} P(H_i)P(D_j|H_i)\ . \tag{5}$$

What a tiny modification to the classical law! In fact, the Born Rule seems to be nothing but a kind of Quantum Law of Total Probability.

Recent work at Perimeter Institute has been devoted much to the SIC existence problem, rewriting the problem in Lie algebraic terms [5], and trying to see how much of the shape of quantum-state space is implied by the very consistency of Eq. (4) [6, 7]. Surprisingly, one can glean quite a bit about the structure of quantum states from the requirement that $Q(D_j)$ always be a proper probability distribution. The talk will end with a list of open problems and avenues for further research.

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