# CLASSIFICATION OF COMPLEX HADAMARD MATRICES

A complex Hadamard matrix is a unitary $N \times N$ matrix with all its matrix elements having absolute value $1/\sqrt{N}$. Thinking of the columns of a matrix as vectors, such a matrix represents a basis that is Mutually Unbiased to the standard basis. They turn up in many other contexts as well; thus one of the two ingredients in Werner's shift-and-multiply construction of unitary operator bases is an arbitrary complex Hadamard matrix.

How many complex Hadamard of a given order exist? We want to classify them up to the equivalence relation $H \sim D_1 P_1 H P_2 D_2$, where $P_i$ are permutation matrices and $D_i$ are diagonal unitaries. This is the natural equivalence if we ask for the set of all pairs of MUB up to unitary equivalence, and indeed in most (but not all) contexts where complex Hadamard matrices arise.

After about a hundred years of work by many authors, the following is known about this problem:

- The Fourier matrix exists for all $N$.
- All complex Hadamard matrices are known for $N \leq 5$. For $N = 2, 3, 5$ only matrices equivalent to the Fourier matrix exist. For $N = 4$ there is a one-parameter family.
- Recently there have been two breakthroughs for the case $N = 6$. A three-parameter family is known, and a four-parameter family is very strongly indicated.
- All real Hadamard matrices have been classified up to $N = 28$.
- All circulant Hadamard matrices (hence all MUB triplets including the Fourier matrix) are known up to $N = 10$. Their number is finite for prime $N$.
- An upper bound for the dimension of all families of complex Hadamard matrices including the Fourier matrix or tensor products of Fourier matrices is known for all $N$. For prime $N$ the Fourier matrix is an isolated matrix, not belonging to any continuous family.

- For $N = 7$ a one-parameter family of complex Hadamard matrices not equivalent to the Fourier matrix exists.

- When $N \leq 16$ is a prime power a maximal family of complex Hadamard matrices including the Fourier matrix is known. A method to construct such families for any $N$ is being investigated.

In my talk I concentrated on the $N = 6$ problem, which has gone from very complicated to nearly solved in four years (as you learned in another talk). This is encouraging!

Rather than give a long list of references, let me refer you to the webpage

http://chaos.if.uj.edu.pl/∼karol/hadamard/

It is kept fully updated by Bruzda, Tadej, and Życzkowski.

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